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Critical mathematical thinking process in definite integral applications: an action, process, object and scheme based on cognitive styles

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ABSTRACT

Higher-order thinking skills are essential in calculus learning. However, research examining how students with different cognitive styles construct mathematical critical thinking processes within the action, process, object and scheme (APOS) framework when solving integral application problems remains limited. Furthermore, the integration of cognitive styles with the APOS framework has not been widely explored. This study investigates differences in students' mathematical critical thinking processes based on cognitive styles using the APOS framework. This study used a qualitative case study design involving 20 students, and four subjects were purposively selected for in-depth analysis, representing field-independent (FI) and field-dependent (FD). Data were collected through critical thinking tests, the group embedded figures test (GEFT), and semi-structured interviews, and analyzed using data reduction, data presentation, and data verification aligned with the APOS stages. The results showed two contrasting reasoning patterns. FI students reached the schema stage with a complete thinking trajectory (understanding the problem, formulating a strategy, performing calculations, and partial evaluation) supported by object-process actions. In contrast, FD students were limited to the action stage with problem-understanding thinking processes. These findings contribute by showing that the development of critical thinking is non-linear and influenced by cognitive style, and emphasize the importance of adaptive learning strategies in calculus.

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1. INTRODUCTION

In 21st-century higher education, critical thinking skills are considered essential competencies. Critical thinking skills can support analytical, evaluative, and reasoning-based decision-making. Numerous studies confirm that critical thinking is not only related to learning outcomes but also reflects the quality of cognitive processes that enable students to construct deep and flexible understanding [1]–[5]. Furthermore, these studies suggest that critical thinking skills can be improved through reflective learning oriented towards problem solving, but this improvement is often partial and uneven across students [2], [6]. Several studies that focus on critical thinking processes reveal that students tend to have difficulty evaluating and justifying solutions in depth, even though they can understand the problem [7]–[9]. This indicates that the main problem is not only in mastering concepts, but also in how students coordinate their thinking processes systematically and reflectively in solving mathematical problems.

In the context of learning calculus, particularly in the application of definite integrals, these problems become increasingly complex. Various studies have shown that integrals are epistemically and cognitively complex concepts because they involve the interrelationships between the concepts of limit, area, and accumulation, and require the ability to coordinate various mathematical representations [10]–[12]. Empirical studies show that students often hold misconceptions about the meaning of integrals, tend to view them as mechanistic procedures, and have difficulty connecting them to geometric and applied contexts [13]–[15]. Furthermore, several studies have confirmed that these difficulties are not only procedural in nature but also related to students' failure to build complete conceptual relationships among various mathematical representations [16]–[18]. In explaining this phenomenon, action, process, object and scheme (APOS) theory provides a framework that emphasizes that conceptual understanding develops through the transformation of mental structures from actions to schemas [17], [19], [20]. However, the development of these mental structures is also influenced by individual characteristics, as research on cognitive styles shows that students with field-independent (FI) and field-dependent (FD) tendencies differ in how they organize information and solve problems [21]–[23]. Thus, difficulties in integral calculus can be understood as the result of the interaction between conceptual complexity, limitations of mental structures, and differences in students' cognitive characteristics.

Previous research has examined various aspects of critical thinking processes, APOS theory, cognitive styles, and integral calculus learning, yet it still shows conceptual fragmentation across studies. Studies on critical thinking processes have mapped the stages of students' thinking in solving mathematical problems, from understanding the problem to evaluating the solution [7], [8]. Several studies also show that, although students can carry out initial stages such as identification and planning, they still experience difficulties at deeper stages of evaluation and reflection [9], [24]. In fact, further analysis indicates that this limitation is related to the less-than-optimal integration of critical thinking processes in the context of solving complex mathematical problems [25].

On the other hand, research based on APOS theory has made important contributions in explaining how students build conceptual understanding through the stages of APOS [26]. In the context of calculus, this approach has been used to analyze understanding of the concepts of derivatives and integrals, including in explaining how students experience difficulties in transforming processes into mathematical objects [26], [27]. Other studies have shown that failure to build integrated mental structures leads to partial and inflexible understanding [17], [28]. In addition, recent research highlights the importance of encapsulating and coordinating mental structures for understanding more complex integral concepts, such as Riemann sums and double integrals [18], [29]. However, the main focus of APOS research is still limited to concept construction and error analysis, and has not explicitly linked it to the process of mathematical critical thinking.

Furthermore, research on cognitive styles shows that individual differences influence how students understand concepts and solve mathematical problems. Individuals with a FI style tend to be better able to organize information analytically and independently [21], while FD individuals rely more on external context in understanding problems [30]. These differences also affect the quality of critical thinking, as certain cognitive styles enable students to be more effective at evaluating and reflecting on solutions [22]. Several other studies also confirm that cognitive style influences problem-solving strategies and the level of errors made by students [31], [32]. However, these studies generally have not systematically linked the role of cognitive style with the development of mental structures in APOS theory.

In addition, studies in the context of integral calculus show that students face various conceptual and procedural difficulties, especially in understanding the meaning of integrals and relating them to real applications [33]. Other research shows that students often experience misconceptions in interpreting integrals as area or as accumulation [34]. This difficulty is also compounded by the inability to coordinate various mathematical representations, such as graphical and symbolic [35]. In several additional studies, it was found that these difficulties were related to weak conceptual understanding and limitations in critical thinking [12], [16], [36]. However, most of this research still stops at the stage of identifying difficulties and has not yet comprehensively explained how the critical thinking process is formed through the APOS mental structure and is influenced by cognitive style.

Based on this synthesis, there appears to be a significant gap in the literature. Research on critical thinking tends to focus on measuring abilities or describing stages of thinking, without in-depth explanations of the underlying cognitive mechanisms [16], [37]. Furthermore, although APOS theory has been widely used to analyze the construction of mathematical concepts, its application remains limited to aspects of conceptual understanding. It has not been explicitly linked to the process of mathematical critical thinking in problem-solving [17], [18], [20]. Then, research on cognitive styles has revealed differences in how students think, but has not provided an integrative explanation of how these differences simultaneously influence the development of mental structures and the quality of critical thinking [22], [23], [25]. In addition, studies on integral calculus are still dominated by the analysis of difficulties and errors, without linking them to the mental construction and dynamics of the critical thinking process in an integrated manner [16]–[18]. Therefore, an approach is needed that integrates critical thinking processes, APOS theory, and cognitive styles in the context of definite integral applications, providing a more comprehensive understanding of how students construct and coordinate knowledge when solving mathematical problems.

To bridge this gap, this study positions mathematical thinking as a dynamic cognitive process influenced by individual abilities, the mental structures underlying knowledge construction, and students' cognitive characteristics. Within this framework, APOS theory is not simply used to explain the development of conceptual understanding, but is viewed as a structural mechanism that allows mapping how students internalize, coordinate, and transform information in the critical thinking process [17], [18], [38]. On the other hand, cognitive style functions as a differential variable that influences how the process unfolds, particularly in organizing information, flexibility of thinking, and problem-solving strategies [21], [39], [40]. Thus, this study integrates three main perspectives of mental structure (APOS), mathematical critical thinking processes, and cognitive styles into a single integrated analytical framework to explain how students construct, evaluate, and reflect on solutions in the context of definite integral applications. This approach is expected to not only expand theoretical understanding of the relationship between knowledge construction and critical thinking, but also provide conceptual contributions in developing a more comprehensive cognitive analysis model in advanced mathematics education.

The limited understanding of how students with different cognitive styles construct and coordinate mental structures when solving problems in definite integral applications is the basis for this research. Therefore, this study aims to analyze in depth students' mathematical critical thinking in solving problems involving definite integrals from the perspectives of APOS theory and cognitive styles. Specifically, this study seeks to identify the stages of students' mathematical critical thinking process, examine the relationship between these stages and the mental structures in APOS theory, and explore how different cognitive styles affect the construction of mental structures and the quality of critical thinking processes. The use of APOS theory in this research is based on its ability to systematically explain the development of students' mental activities in constructing mathematical concepts. Furthermore, this research is expected to provide a more comprehensive understanding of the cognitive mechanisms that encompass mathematical critical thinking, which ultimately becomes an important foundation in designing calculus learning that is adaptive and responsive to students' cognitive characteristics.

2. RESEARCH METHOD

2.1. Research design

This research uses a qualitative approach with a case study design. This design was chosen because it allows for an in-depth exploration of students' mathematical critical thinking processes in solving definite integral application problems through the perspective of APOS theory and cognitive styles. The research focuses not only on the results but also on how students construct and coordinate mental structures during the problem-solving process. The case study is appropriate because it captures phenomena contextually and comprehensively through various data sources, such as written tests and interviews [41], [42]. Thus, this approach is relevant to reveal the cognitive mechanisms underlying mathematical critical thinking more fully.

2.2. Participants

The participants in this study were 20 students in the Mathematics Education Study Program at a University in Indonesia. The participants were selected using a purposive sampling technique. All students were given a mathematical critical thinking test and the group embedded figures test (GEFT) instrument to identify their cognitive styles (FI and FD). Based on these results, four students were selected as the main subjects: two FI and two FD. Selection was based on the clarity and appropriateness of their responses to the written test, enabling an in-depth exploration of the critical thinking process. The limited number of participants was used to obtain rich and in-depth data, not for statistical generalization.

2.3. Research instruments

The research instruments included a mathematical critical thinking test, the GEFT, and semi-structured interviews. The mathematical critical thinking test was used to identify students' abilities and errors in solving problems involving definite integrals, based on indicators of interpretation, analysis strategies, evaluation, and conclusions. The questions were contextualized around the Bono wave phenomenon and validated by experts, with an Aiken's V index of 0.97. The GEFT was used to classify students' cognitive styles as FI or FD based on their scores, with a reliability of 0.82.

Semi-structured interviews were used to delve deeper into students' critical thinking processes and mental structures based on written test responses. The interviews were conducted in a comfortable atmosphere to encourage communication without pressure, allowing students to express the reasons behind their answers openly. Interview questions were flexibly structured based on the students' answer sheets, and the researcher ensured that all questions focused on students' experiences in completing the tests, thus gaining a more comprehensive understanding in accordance with the APOS theoretical framework.

2.4. Data analysis

Data were analyzed qualitatively through the stages of data reduction, data presentation, and conclusion drawing. The analysis began with a review of written test results to identify answer patterns and possible student errors in solving problems. Next, selected subjects were interviewed to confirm and deepen the findings from the written responses. The interviews were recorded and transcribed for further analysis. Data from the tests and interviews were then analyzed in an integrated manner, using indicators of mathematical critical thinking and the APOS theory stages. The analysis was carried out by grouping the data into relevant categories and then interpreting the relationship between critical thinking processes and the construction of students' mental structures. To increase data validity, triangulation was conducted between the test results and interviews to obtain conclusions related to a more comprehensive understanding of students' thinking processes.

3. RESULTS AND DISCUSSION

3.1. Critical mathematical thinking process based on APOS theory of field independent students

After administering the mathematical critical thinking test and the GEFT to 20 students, the researchers identified 6 students with a FI cognitive style. Based on the subject selection criteria for in-depth interviews, the researchers identified two participants who met the requirements and assigned them the codes FI-1 and FI-2. The following section presents an analysis of the critical mathematical thinking processes of subjects FI-1 and FI-2, interpreted through the APOS theoretical framework, in solving problems 1 and 2.

The first step the subjects took was to read, examine, and understand the problem. Both subjects, FI-1 and FI-2, stated the information from the problem and sketched the graph of the given function. This process was clearly evident in the written test responses completed by FI-1 and FI-2.

Based on the excerpt of the written FI-1 and FI-2 responses, as in Figures 1-2, the researchers inferred that both subjects had completed the action stage. The mental structure of interiorization was also evident, as indicated by their ability to interpret symbolic information from the problem into preliminary verbal and mathematical understanding, as well as by their integration of visual and symbolic representations. Furthermore, the subjects demonstrated engagement in the critical thinking stage of understanding and clarifying the problem. To further validate the action stage, interiorization, and the initial critical thinking process, the researchers conducted interviews. Based on the five interview questions, both subjects consistently demonstrated the action stage, the interiorization mental structure, and the understanding and clarifying the problem stage of critical thinking.

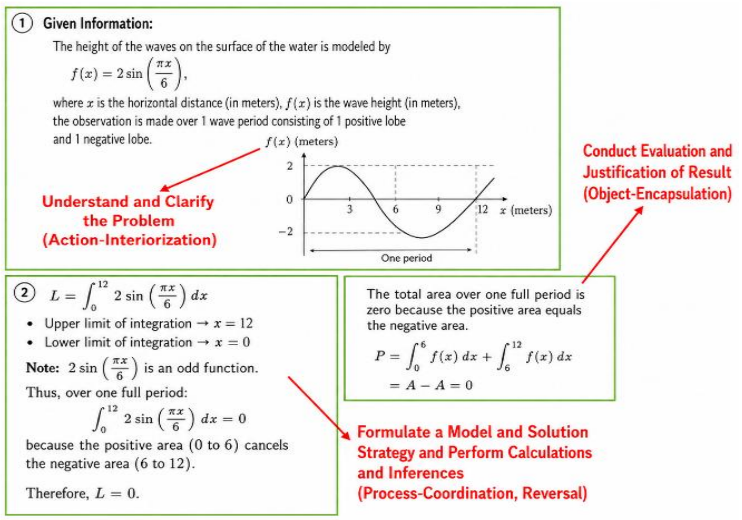

Figure 1. Excerpt of subject FI-1's response

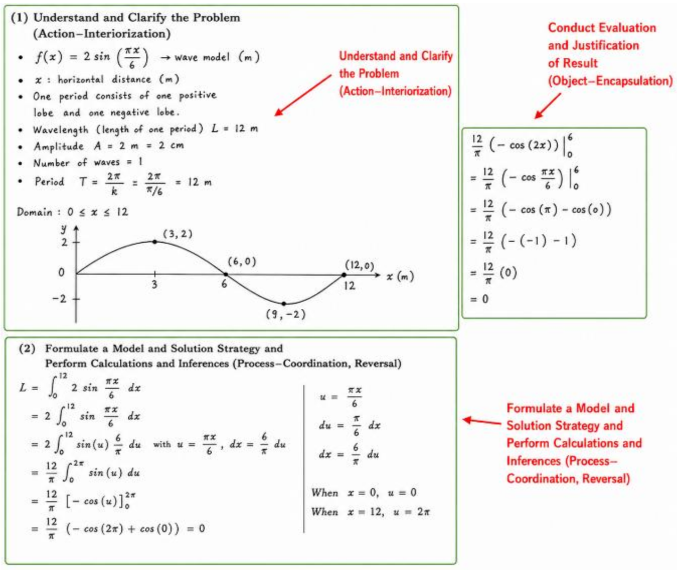

Figure 2. Excerpt of subject FI-2's response

Their engagement in reading the problem, identifying variables and parameters, and visualizing the function graphically primarily demonstrated this. Although their responses to questions 3 and 5 revealed some uncertainty and incomplete understanding, they still demonstrated the action stage, albeit partially, and showed interiorization in a more intuitive form. Overall, both subjects were positioned at the initial phase of conceptual construction, dominated by explicit actions and symbolic interpretation. The next stage, process, the mental structure of coordination, and formulate a model or solution strategy, was marked by the subjects' (FI-1 and FI-2) accurate determination of the integration limits, from $x = 0$ to $x = 6$, which corresponds to half of the sine function's period. Their work demonstrates the ability to coordinate graphical representations with the function's domain and analyze the relevant area spatially. Both subjects formulated the definite integral model $L = \int_0^6 2\, sin\,(\frac{\pi x}{6})dx$ as the mathematical representation of the area.

Furthermore, the researchers identified the mental structure of reversal in their selection of the substitution method $u = \frac{\pi x}{6}$, along with the corresponding adjustment of integration limits. These indicators confirm the presence of the process stage, along with the coordination and reversal structures, and the model formulation and strategy development stage. The following excerpts from the subjects' written responses illustrate these findings. The interview responses further reveal the process stage, the coordination and reversal mental structures, and the stage of formulating a model or solution strategy. Both FI-1 and FI-2 demonstrated substantial achievement in the process stage, the coordination and reversal mental structures, and the formulation of a model or solution strategy stage. The two subjects were able to accurately determine and verify the integration limits by coordinating graphical information, properties of the sine function, and algebraic principles. They also correctly formulated the definite integral model and selected an appropriate substitution strategy aligned with the function's form. This entire process reflects a mature coordination mental structure, in which subjects successfully integrated visual, symbolic, and procedural representations into a coherent and logical mathematical solution.

Next, the object stage, accompanied by the encapsulation mental structure, was evident as the subjects completed the procedure and encapsulated the result as an integral object representing the area of one lobe. Furthermore, the researchers identified the object stage with a de-encapsulation mental structure through the subjects' ability to decompose the total integral object into components based on the function's structure and interval. In-depth interviews with the subjects further clarified the implementation of the object stage, the mental structures of encapsulation and de-encapsulation, and the stage of performing calculations and drawing inferences. Based on the interview excerpts, both subjects reached the object stage and demonstrated the mental structure of encapsulation.

However, the researchers did not observe the de-encapsulation structure, as the subjects were unable to address the validity of the given mathematical statement adequately. Furthermore, the interview responses suggest that both subjects had not fully completed the conduct evaluation and justification of results stage, as they were unable to distinguish between integral calculation as a process and integral calculation as a geometric quantity. The process reflection and generalization stage were also not attained, as the subjects were unable to generalize the idea that the total area is the sum of the absolute values of the regions. Nevertheless, both participants had developed a preliminary schema, although the depth of their generalization requires further development. The thematization of mental structure had not yet fully emerged.

The results of the study indicate that FI students (FI-1 and FI-2) were able to go through almost all stages of the mathematical critical thinking process, from understanding the problem to formulating a model, performing calculations, and drawing conclusions. However, the evaluation and justification stages were not fully optimal. This finding indicates that FI students have strong analytical skills and can independently coordinate various mathematical representations. This ability indicates a more mature cognitive integration than merely procedural. From an APOS perspective, this achievement indicates that FI students have reached the schema stage, in which actions, processes, and objects are fully coordinated [38]. However, the less-than-optimal evaluation indicates that the de-encapsulation and thematization processes are still limited.

These results confirm the findings of [25], [32], who stated that FI students have more systematic and independent critical thinking skills. However, in contrast to Kholid *et al*. [40] who showed that FI's superiority was not always consistent across all critical thinking indicators, this study found that FI still showed limitations at the evaluative stage. Thus, although FI excels in mental structure construction, the strengthening of the reflective aspect is still needed.

3.2. Critical mathematical thinking process based on APOS theory of field dependent students

Subjects FD-1 and FD-2 demonstrated the action stage of critical mathematical thinking along with the mental structure of interiorization. It was evident from their initial step of stating the known information from the problem. However, neither subject carried out these stages comprehensively. Both students failed to fully articulate the given information, provide a graphical representation, and indicate the shaded region whose area they were to calculate. The excerpts from both subjects' responses further clarify these findings.

Subsequently, the researchers conducted interviews to clarify the action stage, the mental structure of interiorization, and the stage of understanding and clarifying the problem demonstrated by the subjects. Based on the interview excerpts, it is evident that subjects FD-1 and FD-2 engaged in the initial APOS stage action involving the mental structure of interiorization, as indicated by their reading of the function and identification of units. However, their ability to progress to subsequent stages of critical thinking remained limited. Both subjects demonstrated a basic understanding of the information in the problem (the function, units, and physical context).

Based on the excerpt of FD-1 and FD-2, as in Figures 3-4, they could not clearly articulate what was being asked (indicating that the stage of understanding and clarifying the problem was incomplete), construct the graph conceptually and accurately (as they drew it intuitively without reference to key points), or determine the integration bounds and the region they needed to calculate (reflecting a failure to coordinate symbolic and geometric representations). Accordingly, the subjects' cognitive pathways remained fragmented. They did not exhibit the coordination mental structure, nor had they entered the process stage of the APOS framework, nor had they formulated a model and solution strategy in critical thinking. In conclusion, both subjects were at an early stage of mathematical information processing and had not yet attained the conceptual integration necessary to solve problems involving definite integrals. Neither demonstrated advancement toward higher-level critical thinking phases nor a more developed cognitive structure.

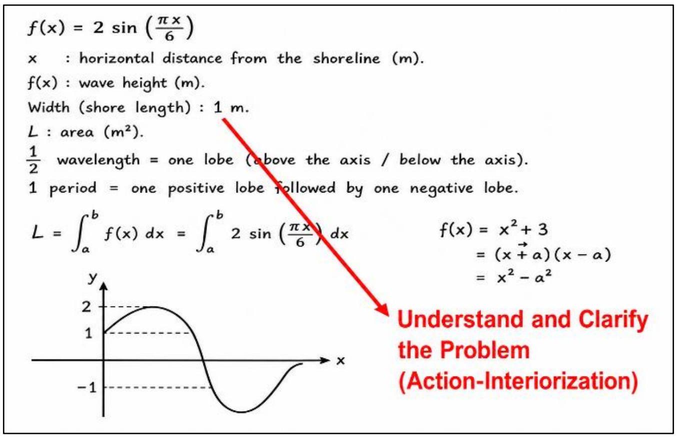

Figure 3. Excerpt of subject FD-1's response

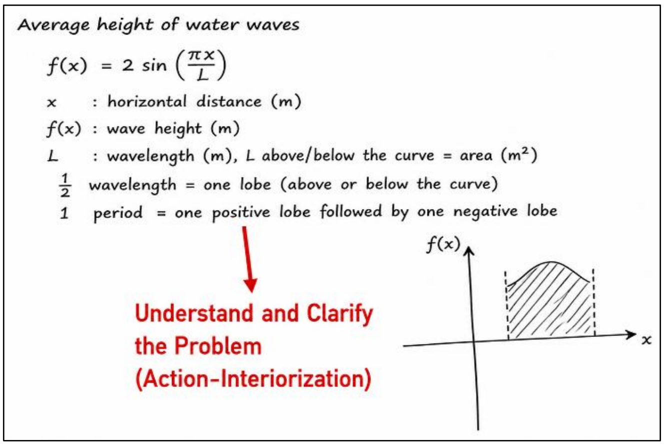

Figure 4. Excerpt of subject FD-2's response

The results of the study indicate that FD students (FD-1 and FD-2) were only able to reach the initial stages of the critical thinking process, namely understanding and clarifying problems, but experienced difficulties in formulating models, making inferences, and drawing conclusions. This finding suggests that FD students tend to rely on available information and are less able to develop strategies independently, leading their thinking processes not to develop systematically. Within the APOS framework, this condition indicates that FD students are still at the action stage (interiorization), where mental activity is procedural and has not yet developed to the process or object stages [38]. This finding aligns with [30], [31], who stated that FD students tend to experience difficulties in analytical thinking and problem-solving. However, unlike the findings of Son *et al.* [43], who showed that FD students can still develop conceptual understanding with certain support, this study shows that without scaffolding, the development of FD mental structures tends to stagnate. This emphasizes the importance of adaptive learning interventions.

3.3. Comparative analysis of critical mathematical thinking processes of field-independent and field-dependent subjects based on APOS theory in the topic of definite integral applications

The findings of this study reinforce and extend previous research showing significant differences between students with FI and FD cognitive styles in mathematical critical thinking processes, particularly in the context of solving problems involving definite integrals. FI students demonstrated superior abilities in modelling problems in definite integral form, such as determining limits of integration based on graphical representations and interpreting the mathematical meaning of area. In contrast, FD students experienced difficulty connecting visual representations with symbolic formulations, resulting in integral models often not appropriate to the problem context.

These results align with previous studies [23], [40], who showed that FI students possess superior critical thinking skills compared to FD students. However, unlike those two studies, which tended to place FI as the superior group overall, this study found that FI students were still suboptimal at the evaluation and justification stages. Specifically, they experienced difficulty in re-verifying the correctness of the integral models they had constructed and in interpreting integral results conceptually. This suggests that FI students' analytical strengths are not yet fully accompanied by mature reflective abilities.

From a cognitive perspective, these differences suggest that cognitive style influences how students construct mathematical knowledge, particularly in understanding the concept of integrals as representations of extent and accumulation. FI students tend to process information analytically and can independently organize various representations, making them more effective at solving problems that require mathematical modeling. This finding is consistent with previous research showing that FI students excel at abstract and complex mathematical tasks [23], [40]. However, this study adds that this advantage is not absolute, particularly at the reflection and evaluation stages, which require higher metacognitive abilities.

When examined through APOS theory, FI students in this study demonstrated relatively complete mental structure development, reaching the APOS stages. They were able to perform integral interiorization procedures, coordinate various representations (graphic, symbolic, and numerical), and encapsulate the integral concept as a mathematical object. However, the incomplete thematization stage indicates that students are not yet fully able to generalize the integral concept as a structure that can be used across contexts. These findings align with APOS-based research, which suggests that students often experience difficulty in achieving advanced mental structures, particularly in abstract calculus concepts [20], [38].

In contrast, FD students in this study tended to be in the action stage, characterized by the mechanical use of procedures without a deep conceptual understanding. They experienced difficulty determining the limits of integration, understanding the relationship between graphs and integrals, and interpreting results for area. This solution demonstrates limitations in the coordination and de-encapsulation processes, resulting in fragmented mental structures. These findings support the research of [30], [31], which showed that FD students experience difficulty in analytical thinking and solving high-level problems. However, unlike Son *et al.* [43], who stated that FD students can still develop through learning support, this study shows that, without structured scaffolding, FD students tend to be unable to move beyond the action stage to the process or object stage.

Furthermore, this study's results indicate that integrating APOS theory and cognitive styles provides a more comprehensive analytical framework for understanding the process of mathematical critical thinking. While previous research tended to examine these two aspects separately, this study integrates them to uncover not only differences in performance but also the cognitive mechanisms underlying the construction of mathematical knowledge. Consistent with Trigueros *et al.* [20], APOS theory is effective in identifying sources of success and difficulty in learning mathematics. This study extends these findings by demonstrating that the development of mental structures in APOS is also influenced by cognitive styles, particularly in the context of definite integral applications that require modeling skills, representational coordination, and conceptual reflection. Overall, this study confirms that cognitive styles influence not only learning outcomes but also the depth and quality of mathematical thinking processes. Therefore, understanding students' cognitive profiles through the integration of APOS theory and cognitive styles provides an important foundation for designing more adaptive calculus learning, particularly in helping students build a more meaningful conceptual understanding of integrals.

4. CONCLUSION

This study concludes that there are significant differences in students' mathematical critical thinking processes, depending on their cognitive styles, when solving problems involving definite integrals. FI students can model problems in definite integral form, determine limits of integration, and coordinate graphical and symbolic representations. However, they are not yet optimal in evaluation, justification, and reflection. Within the APOS framework, they reach the schematic stage but have not yet fully achieved thematization. In contrast, FD students are limited to the action stage and experience difficulty connecting representations and interpreting the meaning of integrals as areas.

Theoretically, this study demonstrates that integrating APOS theory and cognitive styles not only explains performance differences but also reveals the mechanisms underlying the development of mental structures in nonlinear calculus learning. In practice, calculus learning should be designed adaptively, using visual representations to determine limits of integration, problem-based modeling, and reflective assignments to enhance evaluation and generalization. The curriculum needs to emphasize a balance between procedural and conceptual understanding, while the learning design should provide targeted scaffolding for students. Further research is recommended to develop APOS-based learning interventions specifically designed to enhance evaluative and reflective abilities across different cognitive styles.

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AUTHOR CONTRIBUTIONS STATEMENT

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C : **C**onceptualization
M : **M**ethodology
So : **So**ftware
Va : **Va**lidation
Fo : **Fo**rmal analysis
I : **I**nvestigation
R : **R**esources
D : **D**ata Curation
O : Writing - **O**riginal Draft
E : Writing - Review & **E**diting
Vi : **Vi**sualization
Su : **Su**pervision
P : **P**roject administration
Fu : **Fu**nding acquisition

CONFLICT OF INTEREST STATEMENT

Authors state no conflict of interest.

INFORMED CONSENT

We have obtained informed consent from all individuals included in this study.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author, [YLS], upon reasonable request.

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