International Journal of Evaluation and Research in Education (IJERE)
Vol. 13, No. 6, December 2024, pp. 3914~3922
ISSN: 2252-8822, DOI: 10.11591/ijere.v13i6.28206

Comprehension of linear systems with two unknowns in secondary education

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Article Info

Article history:

Received Jul 31, 2023
Revised May 7, 2024
Accepted Jun 21, 2024

Keywords:

Algebra
Linear systems
Mathematics
Secondary education
Understanding

ABSTRACT

Over the past few years, teachers and researchers in mathematics education have emphasized the importance of students acquiring a comprehensive understanding of mathematics. However, during the process of learning mathematics in secondary education, some students encounter difficulties with various algebraic concepts. This article presents the key aspects of a descriptive and exploratory study conducted with two groups of Spanish students in the 4th year of secondary education, using a local model to interpret and evaluate their comprehension of linear systems of two equations with two unknowns. This research offers insights into the phenomenological, epistemological, and cognitive aspects exhibited by the students. In particular, the analysis focuses on the conceptual content of the students' productions. Consistent with similar studies, the results indicate that the students' comprehension of this algebraic knowledge is mainly technical, rote-based and non-meaningful, although some differences exist between the two groups under consideration. Furthermore, the article discusses how these findings may impact the design and implementation of educational proposals focused on improving students' comprehension of this knowledge.

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1. INTRODUCTION

Over the last few decades, there has been a growing concern in the field of mathematics education regarding the importance of understanding. The relevance of meaningful learning in mathematics has been widely recognized [1], [2], among other reasons, because learning with understanding enables students to apply their acquired knowledge to learn new concepts and solve new problems [3]. The phenomenon of comprehending mathematical knowledge is complex due to its multidimensional nature, leading to its examination through different approaches [4], [5].

In the hermeneutical approach, interpretation serves as the core and fundamental requirement for identifying and characterizing comprehension [6]. Within this perspective, the observable records generated during the mathematical activity and their "contextualization" (written mathematical responses, dialogue transcriptions, and recorded actions in videos) represent the main sources for visually expressing understanding.

Furthermore, the semiotic orientation presents comprehension as an essential capacity or competence of students, which translates into interpretable social practices. Interpretation is limited to the exclusive domain of observable mathematical activity and the use of mathematical sign systems through the application of a structural analysis model inspired by linguistics [7]. In the cognitive orientation, mathematical comprehension is conceived as a cognitive phenomenon and, therefore, is not directly observable. Interpretation is carried out by accessing internal cognitive realities with the aid of observation of objectified realizations [8], based on theoretical assumptions regarding the recognized relationship between the subject's mental states and their observable external behavior [9].

The study presented here is part of a research line on the comprehension of specific mathematical knowledge in different areas and educational levels (multiplication algorithm, division algorithm, number systems, and concept of limit). For instance, Gallardo *et al.* [10] studied the mathematical understanding of fractions among pre-service teachers. In this case, our interest lies in diagnosing and assessing the comprehension of systems of two linear equations with two unknowns. This examination is based on a model associated with the epistemology and phenomenology of these systems.

Many studies in mathematics education have focused on the teaching and learning of algebra from various perspectives. For example, Madrid *et al.* [11] study the introduction of algebraic thought in Spain through Spanish mathematics books from the 16th, 17th, and 18th centuries, in particular, they focus on the different methods for solving quadratics equations presented in these books; Duru and Koklu [12] analyze the reading comprehension of mathematical texts and algebraic equations of middle school students; Ayala-Altamirano *et al*. [13] examine the development of algebraic thinking in a group of 25 students age 9-10.

Spanish Real Decreto 217/2022 [14] establishes the organization and minimum curriculum for compulsory secondary education. It structures mathematical knowledge around the concept of mathematical sense, which includes the algebraic sense. This one provides the language through which mathematics is communicated, and its fundamental characteristics are seeing the general in the particular, recognizing patterns and relationships between variables, and expressing them through different representations, as well as modelling mathematical or real-world situations using symbolic expressions. Within the basic concepts included in this algebraic sense are systems of two linear equations with two unknowns.

A system of linear equations with two unknowns consists of two or more equations, each with two unknowns. When the variables in the system have an exponent of one, the system is referred to as linear. Solving such a system means finding the values of each variable that satisfy the equations. Therefore, a solution to a system of equations with two unknowns consists of an ordered pair of numbers that satisfies the system. A linear equation with two unknowns will have infinitely many solutions forming a line (when graphically represented), hence the term “linear equation” [15].

As stated by Istúriz *et al.* [16], equations play a fundamental role in various mathematical concepts throughout compulsory secondary education. Consequently, problems and situations that are solved using systems of two linear equations with two unknowns are frequent and common in the educational system, providing a valuable opportunity to learn basic algebraic concepts and foster problem-solving abilities [17]. To solve a system of two linear equations with two unknowns, secondary education students can use algebraic or geometric methods. According to research, students encounter various cognitive difficulties, some of which can be attributed to their conceptions [18]; the epistemological obstacle of transitioning from arithmetic to algebra [19]; conceptual deficiencies in mathematics [20]. Additionally, Manzanero [21] relies on APOE theory to identify difficulties that students face when studying the concept of the solution of a system of linear equations. Moreover, some studies have explored the relationship that students establish between points on a line and the corresponding solutions of the equation, finding that many students struggle to make this connection [22].

The research problem motivating this study revolves around diagnosing and evaluating students' comprehension of systems of two linear equations as they learn algebra in school. Therefore, the aim of this study is to analyze the understanding of mathematical knowledge, specifically of systems of two linear equations with two unknowns, among Spanish 4th secondary education (ESO) year students. For this purpose, we have chosen an epistemological and phenomenological analysis of systems of two linear equations with two unknowns, based on the studies by Gallardo *et al.* [23].

2. METHOD

The research conducted is descriptive, exploratory, and quantitative in nature, using a written questionnaire with mathematical tasks. The study was carried out during the third trimester of the academic year at a private school in the province of Málaga (Spain). An intentional sample of two groups from the 4th ESO year was chosen:

- Group A: comprised of 24 students in their 4th year of ESO. They have chosen the scientific branch, intend to pursue further studies and, in general, have a good academic performance. They have had exposure to systems of two linear equations with two unknowns twice (during their 3rd year of ESO and in the 1st trimester of their 4th year).

- Group B: comprised of 13 students in their 4th year of ESO. They have chosen the branch oriented towards social sciences and humanities and their academic performance is in general average. They have had exposure to systems of two linear equations with two unknowns once (in their 3rd ESO year).

The questionnaire was designed with an initial set of tasks organized by levels, following the categories proposed in the model of Martín [24]. A set of variables, collectively referred to as task variables, were determined. The type of task variable is a qualitative independent variable with three categories representing three levels of difficulty based on our epistemological classification: technical, analytical, and formal. They describe the necessary elements for diagnosing and evaluating students' comprehension in relation to the analyzed mathematical knowledge. Table 1 illustrates the categories and subcategories within the comprehension/cognitive domain (technical, analytical, and formal), the algebraic knowledge (concept, solution, equivalence, types, and resolution of systems of two linear equations with two unknowns), the different systems of representation (algebraic=A, geometrical=G, numerical=N, and verbal=V), as well as the situations in which pairs of relationships of linear dependence occur simultaneously.

Table 1. Categories and subcategories of the model used

Categories	Algebraic knowledge																		
	Concept of system of two linear equations with two unknowns				Solution of systems of two linear equations with two unknowns				Types of systems of two linear equations with two unknowns				Equivalents systems of two linear equations with two unknowns				Resolution of systems of two linear equations with two unknowns		
	A	G	N	V	A	G	N	V	A	G	N	V	A	G	N	V	A	G	N
Technical	TCA	TCG	TCN	TCV	TSA	TSG	TSN	TSV	TTA	TTG	TTN	TTV	TEA	TEG	TEN	TEV	TRA	TRG	TRN
Analytical	ACA	ACG	ACN	ACV	ASA	ASG	ASN	ASV	ATA	ATG	ATN	ATV	AEA	AEG	AEN	AEV	ARA	ARG	ARN
Formal	FCA	FCG	FCN	FCV	FSA	FSG	FSN	FSV	FTA	FTG	FTN	FTV	FEA	FEG	FEN	FEV	FRA	FRG	FRN

2.1. Technical category

In these tasks, students should demonstrate a mechanical, technical, or routine use of algorithms for solving systems and recognize and identify concepts in its various systems of representation. With this, we aim to establish differences between subjects and obtain interpretable information in terms of comprehension. Specifically, in Figure 1, we provide examples of technical tasks where students are asked to circle the correct answer. These tasks are relatively easy to evaluate, usually admitting only correct or incorrect assessments without further options.

2.2. Analytical category

Non-routine activities that require a reflective use of knowledge by students. They involve analysis and reflection on the structure and external functioning of the algorithm and concepts in its various systems of representation. In particular, we have selected various incomplete tasks and the students are asked to complete them as shown in Figure 2.

2.3. Formal category

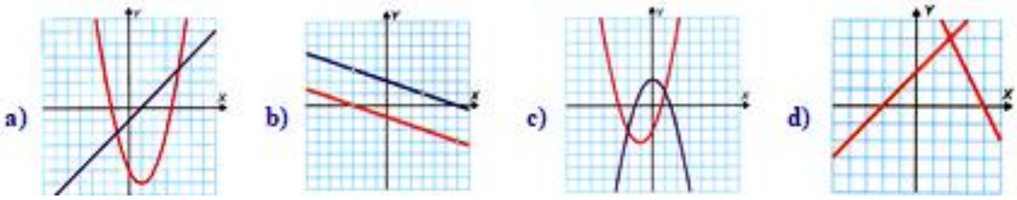
Activities that require a complete knowledge of systems of two linear equations with two unknowns in its different representation systems (definition, properties, and connections). In this type of activity, not only the final answer is evaluated, but also the different resolution processes, strategies used, and arguments presented by each evaluated student. In these tasks students are asked to explain as indicated in the following example as shown in Table 2.

To create the written questionnaire, nineteen tasks were selected for each category (technical, analytical, and formal). They were chosen because they presented favorable conditions for observing the fundamental understanding of systems of two linear equations with two unknowns, as they exclusively focused on the studied mathematical knowledge, but they are not the only questions that can be posed. For the validation of the questionnaire, triangulation was used with experts in mathematics education from the universities of Córdoba, Pontificia de Salamanca and Málaga.

The answers given by the students were considered an observable manifestation of their understanding of systems of two linear equations with two unknowns, despite the limitations involved. The different responses given by the subjects were analyzed and evaluated in terms of comprehension of the studied mathematical knowledge. It was agreed to score the tasks from 0 to 3, following these criteria: i) the student does not do the activity or does it totally incorrectly (0 points); ii) the student answers with reasonable indications of correctness and his answers are in the right direction (1 point); iii) the student gives incomplete or partially correct answers (2 points); and iv) the student gives a correct answer or with minor mistakes (3 points).

Circle the ones that correspond to systems of two linear equations with two unknowns:

a) $\left.\begin{array}{r}2y+x=12\\4y+3x=11\end{array}\right\}$ **b)** $\left.\begin{array}{r}x^2+y^2=25\\2x+y=2\end{array}\right\}$ **c)** $\left.\begin{array}{r}xy=30\\x+3y=21\end{array}\right\}$ **d)** $\left.\begin{array}{r}2x-1=3\\2-y=0\end{array}\right\}$



a)

x	…	-2	-1	0	1	…
y	…	12	11	10	9	…

x	…	-3	-2	-1	0	1	…
y	…	-1	0	1	2	3	…

b)

x	…	-1	0	1	2	4	…
y	…	11	10	9	8	6	…

x	…	-2	-1	1	2	…
y	…	-36	-72	72	36	…

c)

x	…	-2	-1	1	2	…
y	…	4	1	1	4	…

x	…	-4	-2	-1	0	4	…
y	…	33	31	30	29	25	…

d)

x	…	-2	-1	0	1	…
y	…	12	11	10	9	…

x	…	1	1	1	1	…
y	…	3	3	3	3	…

a) Find two numbers such that the sum of the first, and twice the second, is 13, and thrice the first minus the second is four.
b) Find two numbers whose sum is 25, and the square of one of them plus twice the other is 53.
c) The product of two consecutive natural numbers is 132. What are these numbers?
d) Find the age of my brother knowing that he is twice my age, and together we are 13 years old.

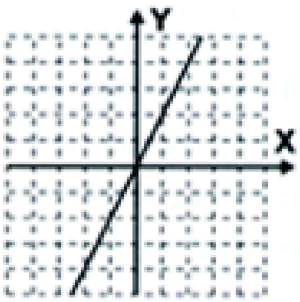
Figure 1. Examples of tasks from the technical category

Complete the following tasks so that each one represents a system of two linear equations with two unknowns:

x	…	-2	-1	0	1	4	…
y	…	12	11	10	9	6	…

x	…						…
y	…						…

$\left.\begin{array}{r}2x+3y=11\\ \dots+\cdots=\cdots\end{array}\right\}$



The difference between two numbers is 8 and …………….
What are these numbers?

Figure 2. Examples of tasks from the analytical category

Table 2. Examples of tasks from the formal category

Explain the concept of system of two linear equations with two unknowns
Using your own words
Using algebraic symbols
Using a graphic representation
Using a numerical table

3. RESULTS

Table 1 shows the average percentages of correct answers obtained for each of the student groups considering the different categories. Regarding the technical category, as shown in Figure 3, associated with algorithmic aspects, it is generally observed that subjects from both groups have good technical knowledge in geometric-type systems (TCG) as well as their solutions (TSG). However, significant differences were found between groups A and B in the concept of equivalent geometric-type (TEG), as well as in the resolution of algebraic-type (TRA) and numerical-type (TRN) systems.

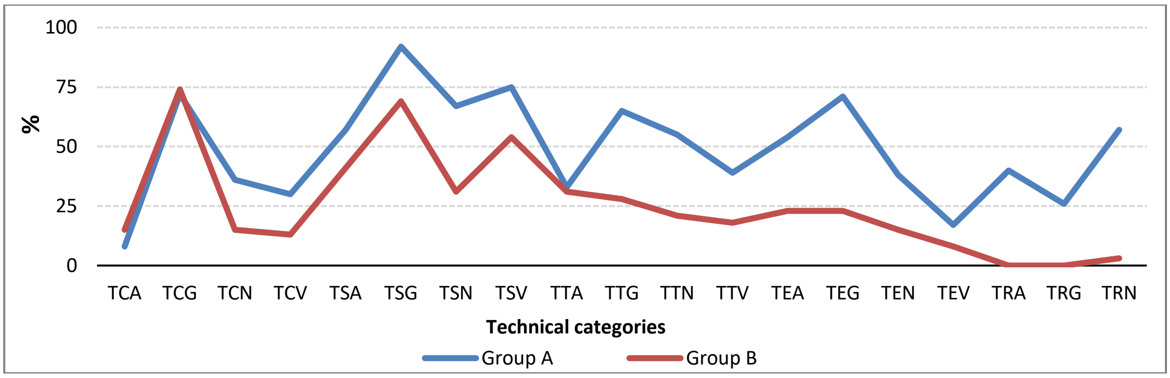


Figure 3. Average percentages of right answers in each group for the technical category

When analyzing the analytical category as shown in Figure 4, a higher percentage of correct responses is observed in group A. Both groups exhibit a similar response pattern, although there are differences in percentage terms. The highest number of correct answers in both groups was found in the concept of algebraic-type solution (ASA). Notable differences were found in the concept of algebraic-type (ACA) and numerical-type (ACN) systems, and the solutions for verbal-type (ASV), numerical-type (ASN), and geometric-type (ASG). Group B obtained no correct answers for ACA, ACN, ACV, ASG, ASV, ASN, ATN, AEN, AEV, ARA, and ARG, so they did not give a correct answer for 63% of the questions.

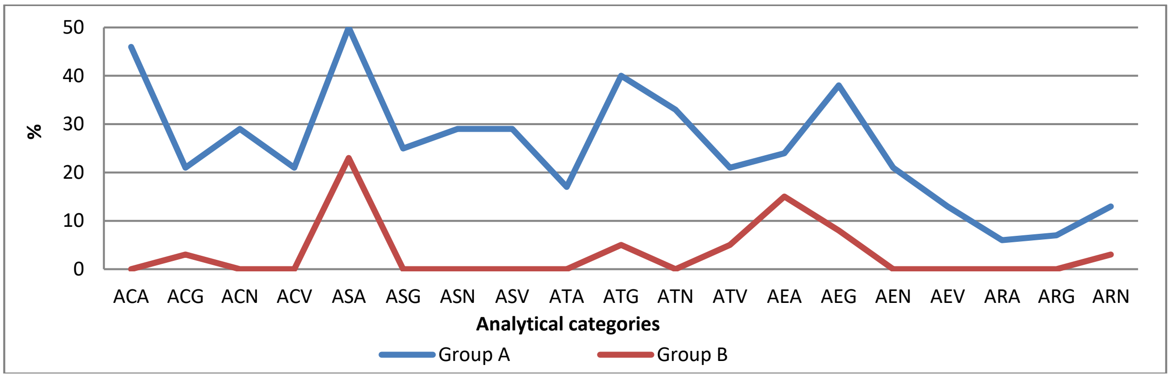


Figure 4. Average percentages of right answers in each group for the analytical category

Considering the formal category as shown in Figure 5, it was found that both groups had the highest percentage of average correct answers in the concept of equivalent verbal-type, with group A being the only one that exceeded 50% of correct answers. Only 21.05% of correct answers were obtained in group B. Regarding the different categories, it is shown that students have a predominantly technical, by rote and instrumental level of understanding of systems of two linear equations with two unknowns. Also, there are similarities in the tasks related to the concept and solution, although with a lower level of accuracy from group B, maybe because they have received less lessons about it or because they have lower interest in the field of mathematics. In equivalent and resolution of systems of two linear equations with two unknowns, the differences are significant maybe due to the different training received by the students according to the academic branch that they have chosen.

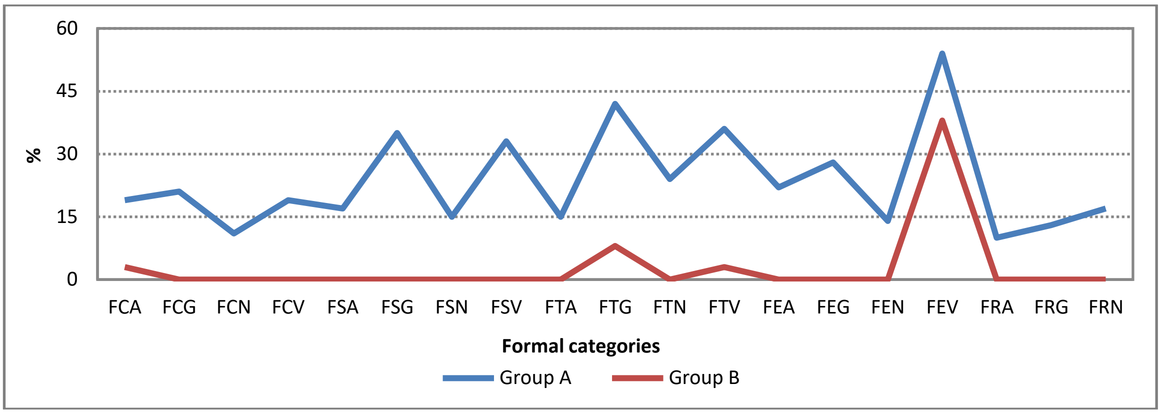


Figure 5. Average percentages of right answers in each group for the formal category

Figure 6 shows frequency polygons of the average percentages of correct responses in each group for the 19 tasks according to the categories of knowledge and system of representation. The results obtained demonstrate that the groups have different levels of understanding but with almost similar tendencies. The tasks related to the solution and the geometric tasks obtained the highest percentage of correct answers. The high percentage of correct answers for the solutions may be because students are more familiar with it due to learning it earlier when solving equations, while for the geometric tasks, the visual component may help comprehension and make them more intuitive.

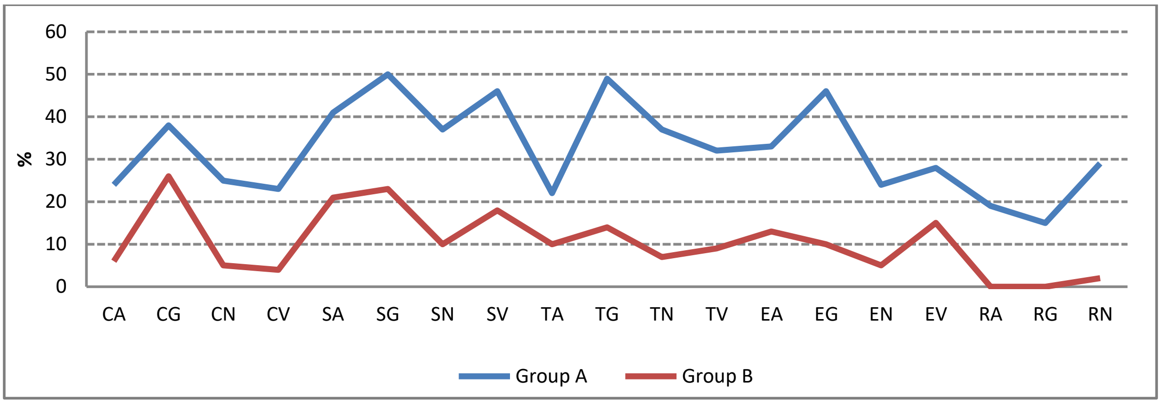


Figure 6. Average percentages of right answers in each group for all tasks

By comparing the frequencies of average percentages of correct responses in each group in the 19 tasks according to the technical, analytical, and formal categories, it is shown that in the technical tasks the students have more right answers, with similar trends of accuracy in both groups. In the technical category, the average percentage of correct responses for group A is 49%, while in the analytical category it is 25%, and in the formal category it is 23%. In contrast, for group B, the average percentage of correct responses is 24%, in the technical category, 3.1% in the analytical category, and 2.8% in the formal category. Finally, when comparing the average of correct responses achieved by both groups (A and B) in the complete questionnaire, group A obtained better results, because they achieved an average of 32% correct responses for the complete questionnaire, while group B obtained an average of 10% correct responses for it.

4. DISCUSSION

The results obtained indicate that students' comprehension of systems of linear equations with two unknowns is in general low and insufficient. Essentially, they have a limited understanding of conceptual aspects, and they mostly rely on instrumental, technical, and rote-based aspects. This situation may be associated with a pragmatic teaching style employed by the teachers. In line with this, Jonsson *et al.* [25] stated that algorithmic learning involves the application of memorized or practiced procedures without reflection on the meaning. Likewise, Edmonds [26] points out that students tend to have difficulties thinking abstractly about algorithms. The students in the sample exhibit a tendency of using and understanding algebraic aspects mainly from a procedural perspective. However, concerning formal aspects, there is a high percentage of positive responses related to algebraic knowledge associated with verbal representations of equivalent systems of equations (FEV).

In terms of technical thinking, particularly with regard to algebraic concepts, students show a good technical knowledge in geometric-type systems (TCG), and this similarity can be observed in both analyzed groups. The situation is similar for solving systems of two linear equations with two unknowns (TSG). These findings are consistent with studies indicating that geometric representation plays an important role in students' transition from recursive calculations to expressing general formulas [27]. Furthermore, in recent years, the promotion and integration of GeoGebra software in mathematics classrooms have helped students internalize geometric representations more frequently and as their preferred mode, compared to other types of representations (algebraic, numeric, tabular, and verbal) [28], [29].

The concepts of algebraic-type (ACA), numerical-type (ACN) systems, and the solutions for verbal-type (ASV), numerical-type (ASN), and geometric-type (ASG) present clear differences both among students and between groups. The existence of such differences has previously been found and highlighted in the study by Barbieri *et al.* [30] regarding the prediction of performance profiles in algebra among secondary education students. While the various types of errors exhibited by the students are concerning, they also present a good opportunity for teachers to attempt to delve deep into both conceptual and procedural understanding. For instance, Hu *et al.* [31] attempted to consider these errors when solving quadratic equations. In this respect, there are many possibilities than can be considered when teaching algebra during secondary education. For example, Jiménez *et al.* [32] proposed the application of the educational escape room and breakout for learning algebra in the third course of secondary education; while Bokhove and Drijvers [33] evaluated the digital tools, which are useful for algebra education. Lastly, Hough *et al.* [34] considered the possibilities of using concept maps to assess teachers' growth in understanding of algebra.

5. CONCLUSION

Results obtained from our sample show that the mathematical knowledge of systems of two linear equations with two unknowns for these 4th year secondary school students was basic and incomplete. The answers provided by the students to the tasks, the detected errors and the employed strategies indicate that most of these students possess instrumental, technical, and rote-based knowledge. They rely on learned techniques and procedures and have a limited and low-level comprehension of the typical representation of systems of two linear equations with two unknowns. The students generally lack awareness of the internal organization of systems of two linear equations with two unknowns, including the principles that characterize them and the justifications for algorithms. Consequently, they struggle to recognize different aspects of these systems.

Based on these results and conclusions, a second empirical study could be proposed in the future to diagnose and evaluate the understanding of a specific sample of students regarding systems of two linear equations with two unknowns. This would help identify and characterize potential associated comprehension states and profiles. Such studies demonstrate that understanding of systems of two linear equations with two unknowns can be significantly improved with appropriate specific didactic approaches. However, these approaches are not commonly developed in secondary schools. This study presents a series of difficulties regarding the algebraic knowledge of secondary school students. It is an opportunity to establish contacts and relationships among mathematics teachers and even mathematics textbook authors, so they can include innovative elements into their teaching and their books that might help overcome or minimize some of these difficulties.

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